2 (Notice that $X : \underline{CAlg} \to \underline{Spc}$ is more data than $X|_{CAlg} \to \underline{Spc}$.) • So $X : \underline{CAlg} \to \underline{Spc}$ s.t. $X|_{CAlg} \simeq Z_0$ (classical scheme) is a der. enh. of Z_0.

One property of $X \in \underline{Sch}$ is that $\forall S_0 \in \underline{Sch}^{aff}$

$X(S)$ is 0-truncated, i.e.

$$\begin{tikzcd} X|_{CAlg} : \underline{CAlg} \arrow[rr] \arrow[dr, dashed] & & \underline{Spc} \\ \underline{CAlg}^{\geq 0} & \underline{Sets} \simeq \underline{Spc}^{\leq 0} \arrow[ur] & \end{tikzcd}$$

factors through this subcategory.

Tools for DAG: - some homotopy thy. (packaged in ∞-cats)
- cotangent complex.

Any (at most) $\mathcal{X} \in \text{PreStk}$ has $T^* \mathcal{X} \in \underline{QCoh}(\mathcal{X})$.

$\underline{QCoh}(\mathcal{X})$ is the der. ∞-cat. of q-coh. sheaves / $\mathcal{X}$.

$\underline{Eg}$: $\underline{QCoh}(pt) = \underline{Vect}$ s.t. $h(\underline{Vect}) = D(k)$
1-categorical shadow of $\underline{Vect}$.

Z derived scheme. $\Rightarrow$ $T^* Z \in \underline{QCoh}(Z)^{\leq 0}$.
i.e. connective cot. complex.
controls def. thy.

$\mathcal{X}$ an n-Artin stack $\Rightarrow$ $\begin{cases} \mathcal{X}(S) \text{ is } (n+1)\text{-truncated.} \\ T^* \mathcal{X} \in QCoh(\mathcal{X})^{\leq n}. \end{cases}$

2 Why derived Algebraic Geometry?

- long (pre-)history that I won't get into. (See. Toën's EMS survey.)

<u>Motivation</u> via examples of usefulness.

I. Good Formal properties.

<u>Hidden smoothness:</u> X curve, Y smooth.
$\rightsquigarrow$ $Maps(X, Y)$ a scheme. (often not smooth)

$T_f Maps(X,Y) \simeq H^0(X, f^*TY)$ at a pt $f: X \to Y$

$\uparrow (\star)\ C^*(X, f^*TY) \simeq \mathbb{T}_f Maps(X,Y)$

To do deformation thy. one normally considers: $\mathbb{T}_f Maps(X,Y)$ tangent complex. $H^0(\mathbb{T})$ agrees. & $H^1(\mathbb{T}_f)$ controls defor.

What happens if X is a surface. ($\dim_{\mathbb{C}} X = 2$).

<u>Thm:</u> (Avramov) X a classical scheme. l.f.t.

$$\Rightarrow \begin{cases} \mathbb{T}_X^* \text{ concentrated in deg. } 0. \\ \text{——————————— s } 0 \,\&\, 1. \\ \mathbb{T}_X^* \text{ is not perfect.} \end{cases}$$

In particular $(\star)$ can not hold. One idea is to give up on $Maps(X, Y)$ altogether or on $\mathbb{T}f$.
Another is: $Maps(X,Y)$ is not really the right object & $C^*(X, f^*TY)$

should guide us to the right object, i.e. a derived enhancement $\mathbb{R}\mathrm{Maps}(X,Y)$ s.t.

$$\pi_f\, \mathbb{R}\mathrm{Maps}(X,Y) \simeq C^*(X, f^*TY).$$

<u>Base change:</u>

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

a pullback in derived schemes $(\underline{\mathrm{Sch}})$.

then

$$f^* \circ g_* \to g'_* \circ (f')^*$$ is an isomorphism of functors $\underline{QCoh}(Y') \to \underline{QCoh}(X)$.

<u>Up-shot:</u> no need for g or f to be flat.

~~However~~ Notice, if they are <u>not</u> & X, Y, Y' are classical schemes

$$X' = X \times^L_Y Y'$$ needs to be derived.

Of course, if either f or g is flat this recovers usual b.c.

<u>II</u>. Geometric Rep. Thy.

Geometric Langlands Conjecture says: X smooth, proper curve. G red.

$$D\text{-mod}(\mathrm{Bun}_G(X)) \simeq \mathrm{Ind}\,\mathrm{Coh}(\mathrm{Loc}_{G^L}(X)).$$

G^L-local syst. i.e. G^L-bundles w/ conn.

G-bundles $/X$.

roughly $\underline{QCoh}(\mathrm{Loc}_{G^L}(X))$.

technical point.

$G = \mathbb{G}_m$ one has $Bun_G(X) = \mathcal{P}ic(X)$ picard stack.

$\mathcal{P}ic(X) \to Pic(X)$ picard sch.

$B\mathbb{G}_m$-torsor.

$D\text{-mod}(\mathcal{P}ic(X))$ has a piece $\simeq D\text{-mod}(B\mathbb{G}_m)$.

Very concretely, $D\text{-mod}(B\mathbb{G}_m) \simeq \underline{Mod}(k[\varepsilon]/(\varepsilon^2))$, $|\varepsilon| = -1$ (cohomologically), i.e.

$$\rightsquigarrow QCoh(\underbrace{Spec(k[\varepsilon]/(\varepsilon^2))}_{\text{this is a derived scheme.}})$$

Thus, $Loc_{G^L}(X)$ needs to be considered as a derived scheme.

<u>Mirković-Riche. linear Koszul duality</u>

V a f.d. vector space $/k$:

$$Mod(\oplus Sym(V)) \simeq QCoh(V^*) \simeq QCoh(pt \times_V pt) = Mod(k \otimes^L_{Sym(V^*)} k)$$

Generalize this two $F_1, F_2 \subset E$ subvector bdles. $/X$ (nice sch.) s.t.

$$Coh_{\mathbb{G}_m}(F_1 \times^L_E F_2) \simeq Coh_{\mathbb{G}_m}(F_1^\perp \times^L_{E^*} F_2^\perp).$$

(and dualities between them).

Riche used this to understand blocks in the category of $U(\mathfrak{g})$-mods where $\mathfrak{g} = Lie(G)$ G conn. simply conn. semisimple alg. grp. $/k$ $\mathrm{char}(k) \gg 0$.

Geometric Affine Hecke algebra.

$G \rightsquigarrow G(K)$ loop group. $(G(K)(\mathbb{C}) = G(\mathbb{C}((t))))$.

$I \subset G$ Iwahori subgroup, (plays a role of Borel subgroup).

Object of interest is $k[I \backslash G(K)/I]$.

(Kazhdan-Lusztig, Ginzburg, Bezrukavnikov).

More geometrically people considered. "$D\text{-mod}(I \backslash G(K)/I)$."

this is equivalent to $\mathrm{Coh}_{G^L}(\tilde{N} \times_{\mathfrak{g}^L} \tilde{N})$

$\tilde{N}$ is the Springer resolution of the subvariety of nilpotent elts. in $\mathfrak{g}^L$ (Langlands dual).

Rk: There are results w/ $I \to I^\circ$ (radical of I) where one can get away with derived fiber product.

III. Enumerative Geometry.

Pert. Obst. theory. Let $M_g(X)$ denote the moduli space of stable maps from a curve of genus g into smooth proper X.

For a number of reasons (unknown to me): one is interested in finding: $\mathcal{E}$ a perfect complex $/ M_g(X)$ w/

$$\varphi : \mathcal{E} \to \mathbb{T}^* M_g(X) \quad \text{s.t.} \quad \mathrm{Cofib}(\varphi) \in \mathrm{QCoh}^{\le -2}_{M_g(X)}$$

I.e. captures degrees $-1, 0$ & maybe above of $\mathcal{M}_g(X)$ but is perfect. Notice from Arinkin's thm we can't expect $\mathcal{E}$ to be $\mathbb{T}^* Z$, Z a classical scheme.

Q: What about Z a stack?

But $\mathbb{R}\mathcal{M}_g(X)$ a derived enhancement of $\mathcal{M}_g(X)$ has.

$\mathbb{T}^*\mathbb{R}\mathcal{M}_g(X)$ w/ such properties.

Fundamental Class: In DAG, given $\mathcal{X}$ a der. stack. & $\mathcal{E} \in \mathcal{QC}oh(\mathcal{X})$ a perfect cmplx one can define:

$$\mathbb{V}(\mathcal{E}) \to \mathcal{X}, \quad \text{roughly} \quad \mathbb{V}(\mathcal{E}) \text{"}:=\text{"} \mathrm{Spec}_{\mathcal{X}}\left(\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}^\vee)\right).$$

Thus, $\mathcal{M}_g(X) \overset{j}{\hookrightarrow} \mathbb{R}\mathcal{M}_g(X)$ allows one to construct a class (in many coh. thy's, e.g. Borel-Moore, K-thy, ...) associated to:

$$\left[\mathcal{M}_g(X) \underset{\mathbb{V}(j^* T^* \mathbb{R}\mathcal{M}_g(X))}{\times} \mathbb{V}(T^*\mathcal{M}_g(X))\right]. \quad \text{(very useful!)}$$

IV. Homotopy theory. DAG offers new insights into homotopy theory.

Étale cohomology of schs: given X a sch an Azumaya alg. $/X$ is a locally free coh. sheaf $/X$ $\mathcal{A}$ s.t. $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{E}nd_X(\mathcal{A})$ is an isom.

$\rightsquigarrow [\mathcal{A}] \in H^2_{\acute{e}t}(X; \mathbb{G}_m)$.

Gabber proved any torsion elt. of $\nearrow$ comes from an $\mathcal{A}$. (X nice).

Toën proved natural derived notion of an Azumaya algebra recovers the whole. (not only torsion) part of $H^2_{et}(X; \mathbb{G}_m)$.

Rational homotopy thy:

[Quillen, Sullivan] $Spc_*^{rat} \subseteq Spc_*$

rational ptd. spaces → $\parallel$

$\{X \in Spc \mid \pi_1(X) = 0, \pi_i(X)$ is a f.d. $\mathbb{Q}$-vect space.$\}$

Thm:

$$Spc_*^{rat} \simeq Lie_{\mathbb{Q}}^{\leq -1}.$$

By "Koszul duality" $CAlg(Vect^{\geq 0})$. is rat. $\ni A^\cdot$ s.t. $H^1(A^\cdot) = 0$.

From DAG perspective we are describing:

$$\mathcal{X} : CAlg^{\leq 0} \to Spc \quad \text{s.t.}$$

$$\mathcal{X}(R) = Hom_{CAlg^{all}}(A, R) \quad \text{for some } A \in CAlg(Vect^{\geq 0})^{rat}$$

$$A \simeq C^*(X; \mathbb{Q}) \quad \& \quad X \to Hom_{CAlg}(C^*(X; \mathbb{Q}), \mathbb{Q})$$

is an equivalence in Spc.

Elliptic cohomology: complicated construct

derived algebraic geometry packages theis data (and give constructions of it) into. $\mathcal{O}^{+}$ an (étale) sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{M}_{1,1}$ the moduli of elliptic curves.

Properties:

- $p : \mathrm{Spec}(R) \xrightarrow{\text{étale}} \mathcal{M}_{1,1} \rightsquigarrow E_p$ an elliptic curve
  $A_p := \mathcal{O}^{+}(\mathrm{Spec}(R))$.

- $H^0(A_p) \simeq R$ & $\mathrm{Spf}\,(A_p^{0}(\mathbb{CP}^\infty)) \simeq \hat{E}_p$

Also $\Gamma(\mathcal{M}_{1,1}, \mathcal{O}+)$ is the long searched for topological modular forms.

V Symplectic geometry.

Vast generalization: instead of considering X a smooth variety. $/\mathbb{C}$.

One can take derived stacks $\mathcal{X}$ locally (almost) of finite type.

n-shifted symplectic structure is (roughly) a duality. (strictly speaking we need more data.)

$$\omega_{\mathcal{X}} : T\mathcal{X} \xrightarrow{\sim} T^{*}\mathcal{X}[n].$$

Here. and are the tangent & cotangent complexes, respectively.

Many constructions in symplectic geometry can be generalized (and sometimes better understood) in the context of shifted symp. geometry.

Symplectic reduction: let G act Hamiltonianlly on X smooth scheme.

$$X/\!\!/G \simeq [X/G] \times_{\mathfrak{g}^*/G} BG.$$

[The map $[X/G] \to \mathfrak{g}^*/G$ is induced by the usual moment map.]

$BG \to \mathfrak{g}^*/G$ is the inclusion of 0 in $\mathfrak{g}^*$.

In s.s.g. one can take a more general elmt of $\mathfrak{g}^*$. (e.g. non regular).

Batalin-Vilkovisky formalism: Often in mathematical physics (i.e. some flavor of field theory) one is interested in.

$f \in \mathcal{O}(M)$ M = space of fields.

$df = 0$ is of interest.

This should really be considered as:

$$\begin{tikzcd} \mathrm{Crit}(f) \arrow[r] \arrow[d] & M \arrow[d, "df"] \\ M \arrow[r, "0"] & T^*M \end{tikzcd}$$

One can actually concretely describe $\mathrm{Crit}(f)$ in many situations of interest roughly as $(\text{"}\mathrm{Sym}(TM[1])\text{"}, -\iota_{df})$ and it carries a (-1)-shifted symplectic structure.
